# Pseudoharmonic oscillator and their associated Gazeau-Klauder coherent states 

Dušan Popov ${ }^{\text {a,* }}$, Vjekoslav Sajfert ${ }^{\text {b }}$, Ioan Zaharie ${ }^{\text {a }}$<br>${ }^{a}$ University "Politehnica" of Timişoara, Department of Technical Physics, B-dul Vasile Pârvan No. 2, 300223 Timişoara, Romania<br>${ }^{\text {b }}$ University of Novi Sad, Technical Faculty "M. Pupin", Djure Djakovića bb, 23000 Zrenjanin, Serbia<br>Received 14 August 2007; received in revised form 21 November 2007<br>Available online 26 February 2008


#### Abstract

In the paper we have constructed and examined the properties of the Gazeau-Klauder coherent states (GK-CSs) for the pseudoharmonic oscillator ( PHO ), one of three possible kinds in order to define the coherent states for this oscillator potential. In the second part, we have examined some nonclassical properties of these states. Our attention has been concentrated on the mixed states (thermal states). The diagonal $P$-representation of the corresponding density operator and some thermal expectations for the quantum canonical ideal gas of pseudoharmonic oscillators have also been examined. Like the CSs for the harmonic oscillator (HO), the GK-CSs for the PHO can be useful in the quantum information theory (QIT). (C) 2008 Elsevier B.V. All rights reserved.


PACS: 03.65.-w; 03.67.-a; 03.65.Ud
Keywords: Coherent state; Density operator; Quantum information; Qubit

## 1. Introduction

As it is well known that the real molecular vibrations are anharmonic, but due to its mathematical advantages, in many problems the harmonic oscillator (HO) model is used. Moreover, in other situations, it is compulsory necessary to use the anharmonic potential models. An anharmonic potential, which also permits an exact mathematical treatment is the so-called "pseudoharmonic oscillator" $(\mathrm{PHO})$ potential, whose effective potential is [1-3]

$$
\begin{equation*}
V_{j}(r)=\frac{m \omega^{2}}{8} r_{0}^{2}\left(\frac{r}{r_{0}}-\frac{r_{0}}{r}\right)^{2}+\frac{\hbar^{2}}{2 m} j(j+1) \frac{1}{r^{2}} \tag{1}
\end{equation*}
$$

where $m$ is the reduced mass, $\omega$ is the angular frequency of the PHO oscillator, while $r_{0}$ is the equilibrium distance between the diatomic molecule nuclei and $j=0,1,2, \ldots$ is the rotational quantum number. This potential can be considered in a certain sense an intermediate potential between the harmonic oscillator potential (an ideal potential)

[^0]and anharmonic potentials (the more realistic potentials). A comparative analysis of potentials HO-3D (3-dimensional harmonic oscillator potential) and PHO is performed in Ref. [3].

Using Molski's techniques [4] (for the Morse oscillator), in a previous paper [5] we have rewritten the PHO effective potential as follows:

$$
\begin{equation*}
V_{j}(r)=\frac{m \omega^{2}}{8} r_{j}^{2}\left(\frac{r}{r_{j}}-\frac{r_{j}}{r}\right)^{2}+\frac{m \omega^{2}}{4}\left(r_{j}^{2}-r_{0}^{2}\right) \tag{2}
\end{equation*}
$$

where the new appearing parameters are

$$
\begin{equation*}
r_{j}=\left[\frac{2 \hbar}{m \omega}\left(\alpha^{2}-\frac{1}{4}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}, \quad \alpha=\left[\left(j+\frac{1}{2}\right)^{2}+\left(\frac{m \omega}{2 \hbar} r_{0}^{2}\right)^{2}\right]^{\frac{1}{2}} . \tag{3}
\end{equation*}
$$

By using the substitution $\omega=2 \omega_{0}$ (the corresponding one-dimensional harmonic oscillator HO-1D has the angular frequency $\omega_{0}$ ) and passing to the dimensionless variable $y=\left(\frac{m \omega_{0}}{h}\right)^{\frac{1}{2}} r$, the corresponding rotational-vibrational Schrödinger equation for the reduced radial function $u_{v}^{\alpha}(r)(v=0,1,2, \ldots, \infty$ is the vibrational quantum number) is [5]

$$
\begin{equation*}
\left[-\frac{1}{2} \frac{\mathrm{~d}^{2}}{\mathrm{~d} y^{2}}+\frac{1}{2} y^{2}+\frac{1}{2}\left(\alpha^{2}-\frac{1}{4}\right) \frac{1}{y^{2}}-(2 v+\alpha+1)\right] u_{v}^{\alpha}(y)=0 . \tag{4}
\end{equation*}
$$

So, the dimensionless Schrödinger equation for the reduced radial function $u_{v}^{\alpha}(r)$ may be written as

$$
\begin{equation*}
H_{\alpha}^{(\mathrm{red})}(y) u_{v}^{\alpha}(y)=e_{v} u_{v}^{\alpha}(y) \tag{5}
\end{equation*}
$$

Here the dimensionless reduced Hamiltonian $H_{\alpha}^{(\text {red })}$ of the PHO appears:

$$
\begin{equation*}
H_{\alpha}^{(\mathrm{red})}(y) \equiv-\frac{1}{2} \frac{\mathrm{~d}^{2}}{\mathrm{~d} y^{2}}+\frac{1}{2} y^{2}+\frac{1}{2}\left(\alpha^{2}-\frac{1}{4}\right) \frac{1}{y^{2}} \tag{6}
\end{equation*}
$$

and $e_{v}=2 v+\alpha+1$ are its dimensionless eigenvalues.
In the previous paper [5] we have demonstrated that the $S U(1,1)$ is the dynamical group associated with the bounded states of the PHO. The Lie algebra corresponding to the group $\operatorname{SU}(1,1)$ is spanned by the three group generators $\left\{K_{1}, K_{2}, K_{3}\right\}$ and the Casimir operator $C_{2}$ for any irreducible representation is the identity times a number:

$$
\begin{equation*}
C_{2}=K_{3}^{2}-K_{1}^{2}-K_{2}^{2}=K_{3}^{2}-\frac{1}{2}\left(K_{+} K_{-}+K_{-} K_{+}\right)=k(k-1) I \tag{7}
\end{equation*}
$$

where $K_{ \pm}=K_{1} \pm \mathrm{i} K_{2}$ are the raising and lowering operators of the group $\operatorname{SU}(1,1)$.
In this context, a representation of $S U(1,1)$ is determined by a single real number $k$ (called the Bargmann index). An algebraic approach to the pseudoharmonic oscillator in two dimensions, as well as the exact resolution of the arbitrary $D$-dimensional Schrödinger equation for this oscillator were performed recently $[6,7]$.

Here, we are interested only in the unitary irreducible representations known as positive discrete series $\mathcal{D}^{+}(k)$, where $k>0$. The corresponding Hilbert space $\mathcal{H}_{k}$ is spanned by the complete orthonormal basis of the number states $|v, k\rangle$ :

$$
\begin{equation*}
\left\langle v, k \mid v^{\prime}, k\right\rangle=\delta_{v v^{\prime}}, \quad \sum_{v=0}^{\infty}|v, k\rangle\langle v, k|=1 . \tag{8}
\end{equation*}
$$

In Ref. [5] we have showed that the following connection between the rotational parameter $\alpha$ (3) and the Bargmann index $k$ exists:

$$
\begin{equation*}
\alpha=2 k-1, \tag{9}
\end{equation*}
$$

i.e. the rotational parameter $\alpha$ plays the role of the Bargmann index. Later in this paper, we will use the $k$-index instead of the $\alpha$-index.

In this way, the dimensionless eigenvalues from Eq. (5) become

$$
\begin{equation*}
e_{v}=2 v+\alpha+1=2(v+k) \tag{10}
\end{equation*}
$$

On the one hand, as we can see from this equation, the energy spectrum of the PHO is similar to the HO-1D energy spectrum, up to a translation in the energy scale. On the other hand, it was showed $[8,9]$ that two specific elements of the positive discrete series $\mathcal{D}^{+}(k)$ of $S U(1,1)$ group unitary irreducible representations (for $k=\frac{1}{4}$ and $k=\frac{3}{4}$ ) reduce the corresponding representation spaces to the Hilbert space of the HO-1D. So, a possible realization of the $S U(1,1)$ group algebra can be given in terms of the creation $\left(a^{+}\right)$and annihilation (a) operators of the HO-1D:

$$
\begin{equation*}
K_{+}=\frac{1}{2}\left(a^{+}\right)^{2}, \quad K_{-}=\frac{1}{2} a^{2}, \quad K_{3}=\frac{1}{2}\left(a^{+} a+\frac{1}{2}\right) \tag{11}
\end{equation*}
$$

For this realization, the Casimir operator becomes $C_{2}=-\frac{3}{16}$, so that $k=\frac{1}{4}$ and $k=\frac{3}{4}$.
The states $\left|v, k=\frac{1}{4}\right\rangle$ of the $S U(1,1)$ group of the PHO correspond to the states $|2 v\rangle$ of the HO-1D, which are the states of the Hamiltonian $H_{0}=2 \hbar \omega K_{3}=\hbar \omega\left(a^{+} a+\frac{1}{2}\right)$ with the energy eigenvalues $E_{v}=\hbar \omega\left(2 v+\frac{1}{2}\right)$. The states $\left|v, k=\frac{3}{4}\right\rangle$ correspond to the states $|2 v+1\rangle$ of the HO-1D, with the energy eigenvalues $E_{v}=\hbar \omega\left(2 v+1+\frac{1}{2}\right)$.

As a consequence of these considerations, we expect that the Gazeau-Klauder coherent states (GK-CSs) which we intend to construct for the PHO will be connected with the standard (Glauber-Klauder-Sudarshan) CSs. This connection will be indicated in the Section "Concluding remarks" in which we will show that the standard CSs for the HO-1D are a limiting case of the GK-CSs for the PHO.

Like the HO-1D (and another few quantum systems, e.g. the Pöschl-Teller potential [10,11,14], as well as Morse [12,13]), the PHO potential admits the construction of three kinds of coherent states: Barut-Girardello, Klauder-Perelomov and Gazeau-Klauder. The first two kinds were the subject of our previous papers, [5,15], while the construction and the examination of the Gazeau-Klauder coherent states (GK-CSs) of the PHO is the aim of the present paper. In this way we intend to continue our previous examination of the coherent states properties of the pseudoharmonic oscillator. Generally, the coherent states are of special importance due to their remarkable mathematical properties and interesting physical applications, especially in quantum optics [16-19] and also in quantum information theory (see, e.g. Ref. [20,21] and the references therein and Ref. [7]).

## 2. GK-CSs of the PHO

In order to associate the usual coherent states with the Hamiltonian problems, Gazeau and Klauder have performed some important modifications in the definition of coherent states [16,17]: the parametrization of the usual coherent state $|z\rangle$ in terms of a single complex number $z$ is extended by replacing $z$ by two independent real numbers $J$ and $\gamma$, so that $J \geq 0$ and $-\infty<\gamma<\infty$, namely $z=\sqrt{J} \exp (-\mathrm{i} \gamma)$. In this way, the new obtained coherent state was denoted by $|J, \gamma\rangle$.

Let us construct the Gazeau-Klauder coherent states (GK-CSs) of the PHO. Corresponding to the original definition [16], these states are

$$
\begin{equation*}
|J, \gamma\rangle=\mathcal{N}(J) \sum_{v=0}^{\infty} \frac{J^{\frac{v}{2}} \mathrm{e}^{-\mathrm{i} \gamma e_{v}}}{\sqrt{\rho(v)}}|v, k\rangle \tag{12}
\end{equation*}
$$

Let us point out that we have preserved the notation for the GK-CSs real parameter $J$ as in the original paper [16] but, in order to avoid the notational confusion with the rotational quantum number, we have denoted the last by the symbol $j$, as it can be observed in Sections 1 and 4.

The quantities $\rho(v)$ (which, as we will see, play the role of the moments of a probability distribution function) are defined as a unique function of $e_{v}$ 's, namely,

$$
\begin{equation*}
\rho(v)=\prod_{j=1}^{v} e_{j}=2^{v} \frac{\Gamma(v+k+1)}{\Gamma(k+1)}=2^{v}(k+1)_{v} \quad \rho(0)=1 \tag{13}
\end{equation*}
$$

where $\Gamma(\ldots)$ is Euler's gamma function, while $(a)_{v}=\frac{\Gamma(a+v)}{\Gamma(a)}$ is the Pochhammer symbol [22].

We note here that the different specific choices of $\rho(v)$ give rise to many different families of coherent states [23-25,17,10].

According to the general prescriptions [16,12] the Gazeau-Klauder coherent state (GK-CS) (12) must be: (a) normalizable; (b) continuous in two labels $J$ and $\gamma$ and must satisfy: (c) the resolution of unity, with a necessarily positive associated measure; (d) the temporal stability condition and (e) the action identity.

The normalization constant $\mathcal{N}\left(|z|^{2}\right)$ can be computed using the normalization condition $\langle J, \gamma \mid J, \gamma\rangle=1$, so that

$$
\begin{equation*}
[\mathcal{N}(J)]^{-2}=\sum_{v=0}^{\infty} \frac{(1)_{v}}{(k+1)_{v}} \frac{\left(\frac{J}{2}\right)^{v}}{v!}={ }_{1} F_{1}\left(1 ; k+1 ; \frac{J}{2}\right) \equiv S_{0}(J) \tag{14}
\end{equation*}
$$

where ${ }_{p} F_{q}(\ldots)$ is the generalized hypergeometric function, particularly ${ }_{1} F_{1}(\ldots)$ is the confluent hypergeometric function [26].

We note that the series (14) determining the normalization constant $\mathcal{N}(J)$ is a convergent series and thus it exists for all values of $J$. This property is very important in order to assure the positivity of the weight function from the integration measure.

Then, the matrix elements of an operator (observable) $A$ which characterize the PHO, in the GK-CSs representation are

$$
\begin{equation*}
\left\langle J^{\prime}, \gamma^{\prime}\right| A|J, \gamma\rangle=\mathcal{N}\left(J^{\prime}\right) \mathcal{N}(J) \sum_{n, v=0}^{\infty} \frac{J^{\prime \frac{n}{2}} J^{\frac{v}{2}}}{\sqrt{\rho(n) \rho(v)}} \mathrm{e}^{-\mathrm{i}\left(\gamma e_{v}-\gamma^{\prime} e_{n}\right)}\langle n, k| A|v, k\rangle . \tag{15}
\end{equation*}
$$

By particularizing the operator $A$, we obtain some interesting properties. So, if $A=I$ (the unity operator), as well as $J^{\prime}=J$ and $\gamma^{\prime}=\gamma$, we recover the normalization constant.

For $A=I$ and using the orthonormality condition (8), we obtain the overlap (the scalar product) of two GK-CSs

$$
\begin{equation*}
\left\langle J^{\prime}, \gamma^{\prime} \mid J, \gamma\right\rangle=\mathcal{N}\left(J^{\prime}\right) \mathcal{N}(J) \sum_{v=0}^{\infty} \frac{\left(J^{\prime} J\right)^{\frac{v}{2}}}{\rho(v)} \mathrm{e}^{-\mathrm{i}\left(\gamma-\gamma^{\prime}\right) e_{v}} \tag{16}
\end{equation*}
$$

The continuity in two labels $J$ and $\gamma$ follows from the continuity of the overlap $\left\langle J^{\prime}, \gamma^{\prime} \mid J, \gamma\right\rangle$ because

$$
\begin{equation*}
\||J, \gamma\rangle-\left|J^{\prime}, \gamma^{\prime}\right\rangle \|^{2}=2\left(1-\operatorname{Re}\left\langle J^{\prime}, \gamma^{\prime} \mid J, \gamma\right\rangle\right) \rightarrow 0, \tag{17}
\end{equation*}
$$

when $\left(J^{\prime}, \gamma^{\prime}\right) \rightarrow(J, \gamma)$.
Generally speaking, resolutions of the identity in terms of a certain set of states are very important because they allow the practical use of these states as a basis in the Hilbert space [27].

In order to prove that the GK-CSs of the PHO resolves the identity, one must find an integration measure $d \mu(J, \gamma)$, so that

$$
\begin{equation*}
\int \mathrm{d} \mu(J, \gamma)|J, \gamma\rangle\langle J, \gamma|=\hat{I}=\sum_{v=0}^{\infty}|v, k\rangle\langle v, k| . \tag{18}
\end{equation*}
$$

The coherent states $|J, \gamma\rangle$ exist only if the radius of convergence $R$ is nonzero [17], and this fact is easy to demonstrate for the case of the PHO:

$$
\begin{equation*}
R=\lim _{v \rightarrow \infty}[\rho(v)]^{\frac{1}{v}}=\lim _{v \rightarrow \infty} \sqrt{\frac{2^{v} \Gamma(v+k+1)}{\Gamma(k+1)}} \rightarrow \infty \tag{19}
\end{equation*}
$$

by using the asymptotic expression of Euler's gamma function [26]

$$
\begin{equation*}
\Gamma(z)=\sqrt{2 \pi} z^{z-\frac{1}{2}} \mathrm{e}^{-z}\left[1+\mathcal{O}\left(\frac{1}{z}\right)\right] \tag{20}
\end{equation*}
$$

If we assume the integration measure $\mathrm{d} \mu(J, \gamma)$ so that [17]

$$
\begin{equation*}
\int(\ldots) \mathrm{d} \mu(J, \gamma)=\frac{1}{2 \pi} \int_{-\pi}^{\pi} \mathrm{d} \gamma \int_{0}^{\infty}(\ldots) k(J) J \mathrm{~d} J \tag{21}
\end{equation*}
$$

we obtain

$$
\begin{equation*}
\int \mathrm{d} \mu(J, \gamma)|J, \gamma\rangle\langle J, \gamma|=\sum_{v=0}^{\infty} \frac{|v, k\rangle\langle v, k|}{\rho(v)} \int_{0}^{\infty} J^{v+1}[\mathcal{N}(J)]^{2} k(J) \mathrm{d} J=1 . \tag{22}
\end{equation*}
$$

After the weight function change

$$
\begin{equation*}
k(J)=\frac{1}{[\mathcal{N}(J)]^{2}} h(J), \tag{23}
\end{equation*}
$$

the Eq. (18) is accomplished if we find the function $h(J)$ satisfying

$$
\begin{equation*}
\int_{0}^{\infty} \mathrm{d} J J^{v+1} h(J)=\rho(v)=2^{v} \frac{\Gamma(v+k+1)}{\Gamma(k+1)} \tag{24}
\end{equation*}
$$

So, it is clear that the function $h(J)$ is the inverse Mellin transform [28] of the function $\rho(v)$. Following the standard method [17] we extend the natural values of $v$ to the complex $s$ so that $v+1 \rightarrow s-1$ and then, using the definition of Meijer's G-function and the Mellin inversion theorem [26,28]

$$
\begin{align*}
& \int_{0}^{\infty} \mathrm{d} x x^{s-1} G_{p, q}^{m, n}\left(\alpha x \left\lvert\, \begin{array}{lllll}
a_{1}, & \ldots, & a_{n}, & a_{n+1}, & \ldots, \\
b_{1}, & \ldots, & b_{m}, & b_{m+1}, & \ldots, \\
b_{q}
\end{array}\right.\right) \\
& =\frac{1}{\alpha^{s}} \frac{\prod_{j=1}^{q} \Gamma\left(b_{j}+s\right) \prod_{j=1}^{n} \Gamma\left(1-a_{j}-s\right)}{\prod_{j=m+1}^{q} \Gamma\left(1-b_{j}-s\right) \prod_{j=n+1}^{p} \Gamma\left(a_{j}+s\right)}, \tag{25}
\end{align*}
$$

we can express the function $h(J)$ in terms of Meijer's G-function

$$
\begin{equation*}
h(J)=\frac{1}{4} \frac{1}{\Gamma(k+1)} G_{01}^{10}\left(\left.\frac{J}{2} \right\rvert\, k-1\right)=\frac{1}{4} \frac{1}{\Gamma(k+1)}\left(\frac{J}{2}\right)^{k-1} \mathrm{e}^{-\frac{J}{2}} . \tag{26}
\end{equation*}
$$

Finally, the weight function (23) is

$$
\begin{equation*}
k(J)=\frac{1}{4} \frac{1}{\Gamma(k+1)}{ }_{1} F_{1}\left(1 ; k+1 ; \frac{J}{2}\right)\left(\frac{J}{2}\right)^{k-1} \mathrm{e}^{-\frac{J}{2}} . \tag{27}
\end{equation*}
$$

Consequently, the integration measure becomes finally

$$
\begin{equation*}
\mathrm{d} \mu(J, \gamma)=\frac{1}{2} \frac{1}{\Gamma(k+1)} \frac{\mathrm{d} \gamma}{2 \pi} \mathrm{~d} J_{1} F_{1}\left(1 ; k+1 ; \frac{J}{2}\right)\left(\frac{J}{2}\right)^{k-1} \mathrm{e}^{-\frac{J}{2}} . \tag{28}
\end{equation*}
$$

As we see, the weight function $k(J)$, as well as the integration measure $\mathrm{d} \mu(J, \gamma)$ are positive functions for all values of label $J$. So, the GK-CSs of the PHO $|J, \gamma\rangle$ form an overcomplete set of states.

Using the definition of Euler's gamma function [26], it is not difficult to prove that this integration measure fulfills the resolution of the identity (18).

The sufficient condition for the solution (27) to be unique is given by the Carleman condition [24,29]: if the solution of the problem (24) exists, then

$$
S \stackrel{\text { def }}{=} \sum_{v=1}^{\infty}[\rho(v)]^{-\frac{1}{2 v}}= \begin{cases}\infty, & \text { the solution is unique }  \tag{29}\\ <\infty, & \text { non-unique solutions exist. }\end{cases}
$$

It is not difficult to prove, using the mathematical analysis test methods (e.g. the logarithmic or d'Alembert test) and also the asymptotic expression of the gamma function (20) which appears in the function $\rho(v)$ defined by Eq. (13), that the sum $S$ diverges. So, the weight function for the GK-CSs of the PHO is unique.

In order to verify the temporal stability condition of the states $|J, \gamma\rangle$, we apply the operator $\exp \left(-\mathrm{i} \omega t H_{\alpha}^{(\mathrm{red})}\right)$ and, using the eigenvalue equation (5), we immediately obtain

$$
\begin{equation*}
\exp \left(-\mathrm{i} \omega t H_{\alpha}^{(\mathrm{red})}\right)|J, \gamma\rangle=\mathcal{N}(J) \sum_{v=0}^{\infty} \frac{J^{\frac{v}{2}} \mathrm{e}^{-\mathrm{i}(\gamma+\omega t) e_{v}}}{\sqrt{\rho(v)}}|v, k\rangle=|J, \gamma+\omega t\rangle . \tag{30}
\end{equation*}
$$

The temporal stability means that, under the chosen dynamics, the temporal evolution of the state $|J, \gamma\rangle$ proceeds to $|J, \gamma+\omega t\rangle$, for an arbitrary fixed positive parameter $\omega$.

The time-dependent GK-CS of the PHO $|J, \gamma ; t\rangle$ can be obtained by acting on the time-independent GK-CS $|J, \gamma ; 0\rangle \equiv|J, \gamma\rangle$ by the time-evolution operator $U(t)=\exp \left(-\mathrm{i} \omega t H_{\alpha}^{(\text {red })}\right)$. Using the temporal stability condition (30), we obtain:

$$
\begin{equation*}
|J, \gamma ; t\rangle=\mathrm{e}^{-\mathrm{i} \omega t H_{\alpha}^{(\mathrm{red})}}|J, \gamma ; 0\rangle \equiv|J, \gamma+\omega t\rangle . \tag{31}
\end{equation*}
$$

As we see, the time dependence has the repercussion only on the label $\gamma$, i.e. the equation which expresses the time dependence of the GK-CS can be written as follows:

$$
\begin{equation*}
|J, \gamma ; t\rangle=|J, \gamma(t)\rangle . \tag{32}
\end{equation*}
$$

The above equation indicates that the GK-CS $|J, \gamma\rangle$ goes over into another GK-CS $|J, \gamma(t)\rangle$ described by the time-dependent label $\gamma$, i.e. $\gamma(t)=\gamma+\omega t$ under the action of the time-evolution operator $U(t)=\exp \left(-\mathrm{i} \omega t H_{\alpha}^{\text {(red) })}\right.$.

If the operator $A$ from Eq. (15) is diagonal in the $|J, \gamma\rangle$-basis, then the sums of the following kind (with $n=0,1,2, \ldots$ ) appear:

$$
\begin{equation*}
S_{n}(J) \equiv \sum_{v=0}^{\infty} \frac{J^{v}}{\rho(v)} v^{n}=\left(J \frac{\mathrm{~d}}{\mathrm{~d} J}\right)^{n} S_{0}(J) \tag{33}
\end{equation*}
$$

which can be expressed as the derivatives of the fundamental sum $S_{0}(J) \equiv{ }_{1} F_{1}\left(1 ; k+1 ; \frac{J}{2}\right)$.
As a consequence, the expectation value of the diagonal operator $A$ in the state $|J, \gamma\rangle$ reads

$$
\begin{equation*}
\langle J, \gamma| A|J, \gamma\rangle=\frac{1}{S_{0}(J)} \sum_{v=0}^{\infty} \frac{J^{v}}{\rho(v)}\langle v, k| A|v, k\rangle . \tag{34}
\end{equation*}
$$

Then, the last condition for the GK-CSs, i.e. the action identity is easy to obtain, using Eq. (5) and (10)

$$
\begin{equation*}
\langle J, \gamma| H_{\alpha}^{(\mathrm{red})}|J, \gamma\rangle=\frac{1}{S_{0}(J)} \sum_{v=0}^{\infty} \frac{J^{v}}{\rho(v)} e_{v}=J \tag{35}
\end{equation*}
$$

as it is pointed out in Ref. [12].
At the end of this section we will point out that the GK-CSs can be useful also in the quantum information theory (QIT).

It is well known that the fundamental unity of quantum information is a qubit (or quantum bit), which is a state (vector) in a two-dimensional Hilbert space of the form:

$$
\begin{equation*}
|\Psi\rangle=a_{0}|0\rangle+a_{1}|1\rangle=\binom{a_{0}}{a_{1}} \tag{36}
\end{equation*}
$$

where $a_{0}$ and $a_{1}$ are the complex numbers and $|0\rangle \equiv\binom{1}{0}$, respectively $|1\rangle \equiv\binom{0}{1}$ are arbitrary base vectors from the state space. The normalization condition requires that $\sum_{n=0}^{1}\left|a_{n}\right|^{2}=1$.

Analogously, a state with many qubits, called a multi-qubit or $N$-qubit can be written as:

$$
\begin{equation*}
|\Psi\rangle=\sum_{n_{1}, n_{2}, \ldots, n_{N}=0,1} a_{n_{1} n_{2} \ldots n_{N}}\left|n_{1} n_{2} \ldots n_{N}\right\rangle \tag{37}
\end{equation*}
$$

where the $2^{N}$ basis vectors are:

$$
\left|n_{1} n_{2} \ldots n_{N}\right\rangle \equiv\left(\begin{array}{c}
n_{1}  \tag{38}\\
n_{2} \\
n_{3} \\
\vdots \\
n_{N}
\end{array}\right), \quad n_{1}, n_{2}, \ldots, n_{N}=0,1
$$

with the normalization relation: $\sum_{n_{1}, n_{2}, \ldots, n_{N}=0,1}\left|a_{n_{1} n_{2} \ldots n_{N}}\right|^{2}=1$.
Evidently that the Fock-basis vectors $|v ; k\rangle$ are also suitable to play the role of basis vectors for a multi-qubit. So, by using the identity operator decomposition (see, Eq. (18), we can immediately express the Fock-basis vectors $|v ; k\rangle$ as an integral over the GK-CSs $|J, \gamma\rangle$ :

$$
\begin{equation*}
|v ; k\rangle=\int \mathrm{d} \mu(J, \gamma) \mathcal{N}(J) \frac{J^{\frac{v}{2}} \mathrm{e}^{+\mathrm{i} \gamma e_{v}}}{\sqrt{\rho(v)}}|J, \gamma\rangle \tag{39}
\end{equation*}
$$

Consequently, a state of $N$-qubits may be written as follows:

$$
\begin{equation*}
|\Psi\rangle=\sum_{n=0}^{2^{N}-1} a_{n} \int \mathrm{~d} \mu(J, \gamma) \mathcal{N}(J) \frac{J^{\frac{v}{2}} \mathrm{e}^{+\mathrm{i} \gamma e_{v}}}{\sqrt{\rho(v)}}|J, \gamma\rangle \equiv \int \mathrm{d} \mu(J, \gamma) f^{*}(J, \gamma)|J, \gamma\rangle \tag{40}
\end{equation*}
$$

where we have used the following notation:

$$
\begin{equation*}
f^{*}(J, \gamma) \equiv \mathcal{N}(J) \frac{J^{\frac{v}{2}} \mathrm{e}^{+\mathrm{i} \gamma e_{v}}}{\sqrt{\rho(v)}} \tag{41}
\end{equation*}
$$

and where, on the one hand, for the simplicity of notation, we have used the index $n$ that signifies in fact the number sequence $n_{1} n_{2} \ldots n_{N} \ldots$, each of them may bring values 0 or 1 and, on the other hand, generally, $f^{*}(J, \gamma)$ is a known complex function [30].

So, in fact we have passed from the discrete-variable (dv) to continuous variable (cv) QIT. The main motivation to deal with continuous variables in QIT is of practical reasons: the essential steps in quantum communication protocols (preparing, unitary manipulating, and measuring entangled quantum states) is achievable in quantum optics by using continuous amplitudes of the quantized electromagnetic field [31].

The above relation can be regarded also as the connection between the quantum information (represented here by $N$-qubit $|\Psi\rangle$ ) and the quantum mechanics (represented by coherent state $|J, \gamma\rangle$ ).

## 3. Nonclassical properties

In order to examine several nonclassical features of the GK-CSs of the PHO, let us calculate the expectations of the power of the number operator $N^{s}$, with $s=1,2, \ldots$. The number operator $N$ is defined as the operator which diagonalize the basis for the number states and so, it follows that

$$
\begin{equation*}
N^{s}|v, k\rangle=v^{s}|v, k\rangle . \tag{42}
\end{equation*}
$$

In the $|J, \gamma\rangle$ state, their expectation values are

$$
\begin{equation*}
\langle J, \gamma| N^{s}|J, \gamma\rangle=\frac{1}{S_{0}(J)} \sum_{v=0}^{\infty} \frac{J^{v}}{\rho(v)} v^{s} \equiv\left\langle N^{s}\right\rangle_{J} \tag{43}
\end{equation*}
$$

which are independent of the label $\gamma$.
The following function

$$
\begin{equation*}
P_{v} \equiv P_{v}(J ; k)=\frac{1}{S_{0}(J)} \frac{J^{v}}{\rho(v)}=\frac{1}{{ }_{1} F_{1}\left(1 ; k+1 ; \frac{J}{2}\right)} \frac{\Gamma(k+1)}{\Gamma(v+k+1)}\left(\frac{J}{2}\right)^{v} \tag{44}
\end{equation*}
$$

is just the weighting distribution function corresponding to the GK-CSs for the PHO.


Fig. 1. Mandel parameter $Q_{J}$ as a function of $J$ for two values of the parameter $k$, i.e. $k=\frac{1}{4}$ and $k=\frac{3}{4}$.
By particularizing the above expression for the first two integer power, i.e. $s=1,2$, after the elementary calculations, we can express the results through the derivatives of the function $\ln S_{0}(J)$ :

$$
\begin{align*}
& \langle N\rangle_{J}=J \frac{\mathrm{~d}}{\mathrm{~d} J} \ln S_{0}(J)=\frac{1}{k+1} \frac{J}{2} \frac{1_{1} F_{1}\left(2 ; k+2 ; \frac{J}{2}\right)}{F_{1}\left(1 ; k+1 ; \frac{J}{2}\right)}  \tag{45}\\
& \left\langle N^{2}\right\rangle_{J}=\left(\langle N\rangle_{J}\right)^{2}+J \frac{\mathrm{~d}}{\mathrm{~d} J}\left(\langle N\rangle_{J}\right) . \tag{46}
\end{align*}
$$

where we have used the derivation formula for the confluent hypergeometric function [26,32]:

$$
\begin{equation*}
\left(\frac{\mathrm{d}}{\mathrm{~d} J}\right)^{n}{ }_{1} F_{1}\left(1 ; k+1 ; \frac{J}{2}\right)=\left(\frac{1}{2}\right)^{n} \frac{\Gamma(n+1) \Gamma(k+1)}{\Gamma(n+k+1)}{ }_{1} F_{1}\left(1+n ; k+1+n ; \frac{J}{2}\right) . \tag{47}
\end{equation*}
$$

These results are useful in order to express the following two characteristic parameters:
(a) The second-order correlation function (the intensity correlation function) [18]:

$$
\begin{equation*}
g_{J}^{2}=\frac{\left\langle N^{2}\right\rangle_{J}-\langle N\rangle_{J}}{\langle N\rangle_{J}^{2}}=1+\frac{J \frac{\mathrm{~d}}{\mathrm{~d} J}\left(\langle N\rangle_{J}\right)-\langle N\rangle_{J}}{\left(\langle N\rangle_{J}\right)^{2}} . \tag{48}
\end{equation*}
$$

(b) The Mandel parameter defined in Refs. $[33,19]$ becomes:

$$
\begin{align*}
Q_{J} & =\frac{\left\langle N^{2}\right\rangle_{J}-\langle N\rangle_{J}^{2}}{\langle N\rangle_{J}}-1=\frac{J \frac{\mathrm{~d}}{\mathrm{~d} J}\left(\langle N\rangle_{J}\right)}{\langle N\rangle_{J}}-1 \\
& =\frac{J}{2}\left[\frac{2}{k+2} \frac{{ }_{1} F_{1}\left(3 ; k+3 ; \frac{J}{2}\right)}{F_{1}\left(2 ; k+2 ; \frac{J}{2}\right)}-\frac{1}{k+1} \frac{{ }_{1} F_{1}\left(2 ; k+2 ; \frac{J}{2}\right)}{F_{1}\left(1 ; k+1 ; \frac{J}{2}\right)}\right] . \tag{49}
\end{align*}
$$

These two parameters, as functions of the label $J$ (they are independent of $\gamma$ ) provide the information about the inherent statistical properties of the GK-CSs $|J, \gamma\rangle$ and allows one to distinguish between the sub-Poissonian, the Poissonian or supra-Poissonian statistics of these CSs.

Generally speaking, the state for which $Q_{J}>0$ (or $g_{J}^{2}>1$ ) is called super-Poissonian (bunching effect), if $Q_{J}=0\left(\right.$ or $\left.g_{J}^{2}=1\right)$ the state is called Poissonian, while a state for which $Q_{J}<0\left(\right.$ or $\left.g_{J}^{2}<1\right)$ is called subPoissonian (antibunching effect).

It is easily checked that $Q_{J} \geq 0$ for all $J \geq 0$ which shows that the GK-CSs for PHO are super-Poissonian. Numerical study confirms that this is indeed true, as displayed in Fig. 1.

This behavior of the GK-CSs for PHO is similar to the combinatorial coherent states built on sequences of integers originating from the solution of the boson normal ordering problem [34].


Fig. 2. The weighting distribution $P_{v}$ of the GK-CSs for two values of parameter $k\left(\frac{1}{4}\right.$ and $\left.\frac{3}{4}\right)$, corresponding to relatively small $J$ (a) and large $J$ (b), as function of $v$.

For small values of $J$, the asymptotic behavior of the confluent hypergeometric function ${ }_{1} F_{1}\left(n ; k+n ; \frac{J}{2}\right)$ is given immediately by the first terms of this series, while for large $J$ we have [35,32]:

$$
\begin{equation*}
{ }_{1} F_{1}\left(n ; k+n ; \frac{J}{2}\right)=\frac{\Gamma(k+1)}{\Gamma(n)} \mathrm{e}^{\frac{J}{2}}\left(\frac{J}{2}\right)^{-k}\left[1+\mathcal{O}\left(\frac{2}{J}\right)\right], \quad J \rightarrow \infty \tag{50}
\end{equation*}
$$

which confirm the behavior of the Mandel parameter for small, respectively large values of $J$, i.e. $Q_{J} \simeq 0$, as it is showed in Fig. 1.

The case of large $J$ is of special interest, because for these values the GK-CSs have the Poissonian statistics, because $Q_{J} \simeq 0$. By using Eqs. (45) and (50) we obtain, on the one hand, that the mean value of the number operator $N$ for large $J$ is

$$
\begin{equation*}
\langle N\rangle_{J}=\frac{J}{2} \tag{51}
\end{equation*}
$$

On the other hand, the weight distribution function $P_{v}(44)$ becomes, for large $J$ :

$$
\begin{equation*}
P_{v}=\frac{1}{\Gamma(v+k+1)}\left(\frac{J}{2}\right)^{v+k} \mathrm{e}^{-\frac{J}{2}} \tag{52}
\end{equation*}
$$

Inserting the asymptotic value for $\langle N\rangle_{J}$ Eq. (51), we obtain:

$$
\begin{equation*}
P_{v}=\frac{1}{\Gamma(v+k+1)}\left(\langle N\rangle_{J}\right)^{v+k} \mathrm{e}^{-\langle N\rangle_{J}} . \tag{53}
\end{equation*}
$$

This is just the expression of a Gamma distribution function, with an inverse scale parameter equal to unity [36]. Their behavior as function of energy quantum number $v$ is presented in Fig. 2.

It is fruity to observe that for $k=0$ this weighing distribution function becomes to be Poissonian:

$$
\begin{equation*}
P_{v}=\frac{1}{v!}\left(\langle N\rangle_{J}\right)^{v} \mathrm{e}^{-\langle N\rangle_{J}} . \tag{54}
\end{equation*}
$$

In order to compare the statistics of different kinds of CSs related to the PHO, we have computed the Mandel parameter by using the results obtained in two of our previous papers, for the Barut-Girardello coherent states for PHO (i.e. BG-CSs) [5] and for the Klauder-Perelomov coherent states for PHO (i.e. KP-CSs) [15].

The BG-CSs for PHO are [5]:

$$
\begin{equation*}
|z, k\rangle_{B G}=\sqrt{\frac{|z|^{2 k-1}}{I_{2 k-1}(2|z|)}} \sum_{v=0}^{\infty} \frac{z^{v}}{\sqrt{\Gamma(v+1) \Gamma(v+2 k)}}|v, k\rangle \tag{55}
\end{equation*}
$$



Fig. 3. The dependence of the Mandel parameter of two kinds of CSs for the PHO as functions of $J\left(=|z|^{2}\right)$ : for the GK-CSs (two curves from the top of figure), for $k=\frac{1}{4}$ and $k=\frac{3}{4}$, respectively for the BG-CSs (two below curves) for $k \simeq \frac{1}{4}$ and $k \simeq \frac{3}{4}$.

Table 1

| CSs | Mandel parameter | $J=\|z\|^{2}$ small | $J=\|z\|^{2}$ large |
| :--- | :--- | :--- | :--- |
| BG-CSs [5] | Eq. (56) | Sub-Poissonian | Poissonian |
| GK-CSs [this paper] | Eq. (49) | Supra-Poissonian | Poissonian |
| KP-CSs [15] | Eq. (58) | Supra-Poissonian | $/$ |

where $I_{2 k-1}(2|z|)$ are the modified Bessel functions, so that the corresponding Mandel parameter is

$$
\begin{equation*}
Q_{z, k}^{(B G)}=|z|\left[\frac{I_{2 k+1}(2|z|)}{I_{2 k}(2|z|)}-\frac{I_{2 k}(2|z|)}{I_{2 k-1}(2|z|)}\right] . \tag{56}
\end{equation*}
$$

By using the asymptotic expressions for the modified Bessel function [26,28], we can deduce that the corresponding BG-CSs for the PHO have sub-Poissonian statistics $\left(Q_{z, k}^{(B G)}<0\right)$ for $|z|$ small, while for large $|z|$ these states tend to have Poissonian statistics $\left(Q_{z, k}^{(B G)}=0\right)$. In Fig. 3 we comparatively have showed the dependence of the Mandel parameter of two kinds of CSs for the PHO as functions of $J\left(=|z|^{2}\right)$ : for the GK-CSs (two curves from the top of figure), for $k=\frac{1}{4}$ and $k=\frac{3}{4}$, respectively for the BG-CSs (two below curves) for $k \simeq \frac{1}{4}$ and $k \simeq \frac{3}{4}$.

This behavior is also in accordance with that obtained in Ref. [10] for the temporally stable GK-CSs for infinite well and Pöschl-Teller potentials.

The KP-CSs for PHO have been built in [15]:

$$
\begin{equation*}
|z, k\rangle_{K P}=\left(1-|z|^{2}\right)^{k} \sum_{v=0}^{\infty} \frac{z^{v}}{\sqrt{\frac{\Gamma(v+1) \Gamma(2 k)}{\Gamma(v+2 k)}}}|v, k\rangle \tag{57}
\end{equation*}
$$

being defined only on a disk with the radius $|z|=1$.
As a consequence, the corresponding Mandel parameter is

$$
\begin{equation*}
Q_{z, k}^{(K P)}=\frac{|z|^{2}}{1-|z|^{2}} \tag{58}
\end{equation*}
$$

and it is, evidently, positive, i.e. the corresponding KP-CSs are supra-Poissonian, for all values of $J=|z|^{2}<1$.
By concluding, the statistical behavior of different CSs associated to the PHO is presented in Table 1.
We see from the Table that the relevant comparation must be made only between two of three kinds of CSs: BGCSs and GK-CSs (the KP-CSs being defined only on the part of the complex plane $z$ ). These two kinds of states have opposite statistics: sub-Poissonian (for the BG-CSs) versus supra-Poissonian (for the GK-CSs). But, for the large values of the parameter $J=|z|^{2}$, both kinds of these CSs tend to involve as Poissonian.

## 4. Statistical properties

As we have proceeded in the previous papers [5,14] and also in order to have a comparison basis, in connection with the statistical properties of the GK-CSs of the PHO, we consider a quantum canonical gas of PHOs in thermodynamic equilibrium with a thermostat at temperature $T=\left(k_{\mathrm{B}} \beta\right)^{-1}$, where $k_{\mathrm{B}}$ is Boltzmann's constant and $\beta$ is the well-known temperature parameter.

The corresponding normalized density operator for a fixed rotational quantum number $j$ (or, equivalently, via Eqs. (3) and (9), for a fixed number $k$ ), is then

$$
\begin{equation*}
\rho_{j} \equiv \rho_{k}=\frac{1}{Z_{k}} \sum_{v=0}^{\infty} \mathrm{e}^{-\beta E_{v k}}|v, k\rangle\langle v, k| \tag{59}
\end{equation*}
$$

where $Z_{k}$ is the normalization constant, i.e the partition function for a certain rotational state $j$ and $E_{v k}$ is the eigenvalue of the effective potential (1) [5]

$$
\begin{equation*}
E_{v k}=2 \hbar \omega_{0}(v+k)-m \omega_{0}^{2} r_{0}^{2} \tag{60}
\end{equation*}
$$

The $Q$-function (Husimi's $Q$-function), i.e. the diagonal elements of the density operator in the representation of the GK-CSs, is

$$
\begin{equation*}
\langle J, \gamma| \rho_{k}|J, \gamma\rangle=\frac{1}{Z_{k}} \sum_{v=0}^{\infty} \mathrm{e}^{-\beta E_{v k}}|\langle J, \gamma \mid v, k\rangle|^{2}=\frac{1}{Z_{k}} \frac{1}{S_{0}(J)} \sum_{v=0}^{\infty} \mathrm{e}^{-\beta E_{v k}} \frac{J^{v}}{\rho(v)} . \tag{61}
\end{equation*}
$$

This function can be expressed through the thermal mean occupancy for a one-dimensional harmonic oscillator (HO-1D) with the angular frequency $\omega=2 \omega_{0}$, i.e.

$$
\begin{equation*}
\bar{n}=\frac{1}{\mathrm{e}^{2 \beta \hbar \omega_{0}}-1} \tag{62}
\end{equation*}
$$

So, we obtain

$$
\begin{align*}
\langle J, \gamma| \rho_{k}|J, \gamma\rangle & =\mathrm{e}^{\beta m \omega_{0}^{2} r_{0}^{2}} \frac{\left(\frac{\bar{n}}{\bar{n}+1}\right)^{k}}{Z_{k}} \frac{1}{S_{0}(J)} \sum_{v=0}^{\infty} \frac{\left(J \frac{\bar{n}}{\bar{n}+1}\right)^{v}}{\rho(v)} \\
& \equiv \mathrm{e}^{\beta m \omega_{0}^{2} r_{0}^{2}} \frac{\left(\frac{\bar{n}}{\bar{n}+1}\right)^{k}}{Z_{k}} \frac{S_{0}\left(J_{\overline{\bar{n}}}^{\bar{n}+1}\right)}{S_{0}(J)} \tag{63}
\end{align*}
$$

By normalizing the density operator to unity, i.e.,

$$
\begin{equation*}
\operatorname{Tr} \rho_{k}=\int \mathrm{d} \mu(J, \gamma)\langle J, \gamma| \rho_{k}|J, \gamma\rangle=1 \tag{64}
\end{equation*}
$$

we obtain the correct expression for the partition function of the PHO (for the fixed rotational state $j$ ) [5]:

$$
\begin{equation*}
Z_{k}=\mathrm{e}^{\beta m \omega_{0}^{2} r_{0}^{2}} \mathrm{e}^{-\beta \hbar \omega_{0}(2 k-1)} \frac{1}{2 \sinh \beta \hbar \omega_{0}}=\mathrm{e}^{\beta m \omega_{0}^{2} r_{0}^{2}}\left(\frac{\bar{n}}{\bar{n}+1}\right)^{k}(\bar{n}+1) . \tag{65}
\end{equation*}
$$

This is, of course, the same as the expression that we have obtained previously [37] Eq. (41) by using the trace of the PHO density matrix in the position representation. It is easy to observe that the way using GK-CSs is much simpler.

Using the expression of the partition function, finally we obtain the following expression for the diagonal elements of the density matrix (63):

$$
\begin{equation*}
\langle J, \gamma| \rho_{k}|J, \gamma\rangle=\frac{1}{\bar{n}+1} \frac{S_{0}\left(J_{\overline{\bar{n}}+1}\right)}{S_{0}(J)}=\frac{1}{\bar{n}+1} \frac{{ }_{1} F_{1}\left(1 ; k+1 ; \frac{J}{2} \frac{\bar{n}}{\bar{n}+1}\right)}{{ }_{1} F_{1}\left(1 ; k+1 ; \frac{J}{2}\right)} . \tag{66}
\end{equation*}
$$

Let us now perform the diagonal expansion of the density operator in the GK-CSs representation:

$$
\begin{equation*}
\rho_{k}=\int \mathrm{d} \mu(J, \gamma)|J, \gamma\rangle P_{k}^{(G K)}(J)\langle J, \gamma| . \tag{67}
\end{equation*}
$$

In order to find the quasi-probability distribution function $P_{k}^{(G K)}(J)$ (the $P$-distribution or $P$-function), from the diagonal expansion of the density operator in the GK-CSs representation, we observe that the equation

$$
\begin{equation*}
\langle f| \rho_{k}|g\rangle=\int \mathrm{d} \mu(J, \gamma)\langle f \mid J, \gamma\rangle P_{k}^{(G K)}(J)\langle J, \gamma \mid g\rangle \tag{68}
\end{equation*}
$$

must be fulfilled for any arbitrary vectors $\langle f|$ and $|g\rangle$ from the Hilbert space (or for the basis vectors $|J, \gamma\rangle$ or $|v, k\rangle$ ). What is more, from the trace condition (64) and the scalar product (the overlap) (16), it follows that the $P$-function satisfies the normalization condition

$$
\begin{equation*}
\int \mathrm{d} \mu(J, \gamma) P_{k}^{(G K)}(J)=1 . \tag{69}
\end{equation*}
$$

The left-hand side of Eq. (68) is

$$
\begin{equation*}
\mathrm{LHS} \equiv \frac{1}{Z_{k}} \sum_{v=0}^{\infty} \mathrm{e}^{-\beta E_{v k}}\langle f \mid v, k\rangle\langle v, k \mid g\rangle \tag{70}
\end{equation*}
$$

while, in the mean time, after the angular integration,

$$
\begin{equation*}
\frac{1}{2 \pi} \int_{-\pi}^{\pi} \mathrm{d} \gamma \mathrm{e}^{-\mathrm{i}\left(e_{n}-e_{v}\right) \gamma}=\delta_{n v} \tag{71}
\end{equation*}
$$

the right-hand side becomes

$$
\begin{equation*}
\text { RHS } \equiv \frac{1}{\Gamma(k+1)} \sum_{v=0}^{\infty} \frac{\langle f \mid v, k\rangle\langle v, k \mid g\rangle}{\rho(v)} 2^{v} \int_{0}^{\infty} \mathrm{d}\left(\frac{J}{2}\right)\left(\frac{J}{2}\right)^{k+v} \mathrm{e}^{-\frac{J}{2}} P_{k}^{(G K)}(J) . \tag{72}
\end{equation*}
$$

Comparing the LHS and the RHS, the above integral must be

$$
\begin{equation*}
\int_{0}^{\infty} \mathrm{d}\left(\frac{J}{2}\right)\left(\frac{J}{2}\right)^{k+v} \mathrm{e}^{-\frac{J}{2}} P_{k}^{(G K)}(J)=\frac{2^{-v}}{Z_{k}} \mathrm{e}^{-\beta E_{v k}} \Gamma(k+1) \rho(v) . \tag{73}
\end{equation*}
$$

Using the Eqs. (60), (65) and (13), performing the variable change $\frac{J}{2}=x$ and the function change

$$
\begin{equation*}
P_{k}^{(G K)}(J)=\frac{1}{\bar{n}+1} \mathrm{e}^{\frac{J}{2}} h_{k}\left(\frac{J}{2}\right) \tag{74}
\end{equation*}
$$

we obtain the following equation, representing a Stieltjes moment problem, of the same kind as for the Eq. (24):

$$
\begin{equation*}
\int_{0}^{\infty} \mathrm{d} x x^{v+k} h_{k}(x)=\frac{1}{\left(\frac{\bar{n}+1}{\bar{n}}\right)^{v}} \Gamma(v+k+1) \tag{75}
\end{equation*}
$$

Following the standard procedure [17], i.e. extending the natural variables $v$ to the complex $s$ so that $v+k \rightarrow s-1$ and using the definition of Meijer's G-functions and the Mellin inversion theorem [28], we obtain

$$
\begin{equation*}
h_{k}(x)=\left(\frac{\bar{n}+1}{\bar{n}}\right)^{k+1} G_{01}^{10}\left(\left.\frac{\bar{n}+1}{\bar{n}} x \right\rvert\, 0\right)=\left(\frac{\bar{n}+1}{\bar{n}}\right)^{k+1} \mathrm{e}^{-\frac{\bar{n}+1}{\bar{n}} x} . \tag{76}
\end{equation*}
$$

Finally, the $P$-distribution function of the PHO density operator takes the form

$$
\begin{equation*}
P_{k}^{(G K)}(J)=\frac{1}{\bar{n}}\left(\frac{\bar{n}+1}{\bar{n}}\right)^{k} \mathrm{e}^{-\frac{1}{\bar{n}} \frac{J}{2}}=\left(\mathrm{e}^{2 \beta \hbar \omega_{0}}-1\right)\left(\mathrm{e}^{2 \beta \hbar \omega_{0}}\right)^{k} \mathrm{e}^{-\left(\mathrm{e}^{\left.2 \beta \hbar \omega_{0}-1\right) \frac{J}{2}} .\right.} \tag{77}
\end{equation*}
$$

With the help of the above obtained expression for the integration measure Eq. (28) and the expression for the $P$-distribution function (Eq. (77)), after the angular integration, the Eq. (69) becomes

$$
\begin{equation*}
\frac{1}{\bar{n}}\left(\frac{\bar{n}+1}{\bar{n}}\right)^{k} \frac{1}{\Gamma(k+1)} \int_{0}^{\infty} \mathrm{d}\left(\frac{J}{2}\right)\left(\frac{J}{2}\right)^{k} \mathrm{e}_{1}^{-\frac{\bar{n}+1}{\bar{n}} \frac{J}{2}} F_{1}\left(1 ; k+1 ; \frac{J}{2}\right)=1 . \tag{78}
\end{equation*}
$$

The above integral $\equiv I$ is of the following kind [26]

$$
\begin{align*}
& \int_{0}^{\infty} \mathrm{d} x x^{\sigma-1} \mathrm{e}^{-\mu x}{ }_{m} F_{n}\left[a_{1}, \ldots, a_{m} ; b_{1}, \ldots, b_{n} ;(\lambda x)^{s}\right] \\
& \quad=\frac{\Gamma(\sigma)}{\mu^{\sigma}}{ }_{m+s} F_{n}\left[a_{1}, \ldots, a_{m}, \frac{\sigma}{s}, \frac{\sigma+1}{s}, \ldots, \frac{\sigma+s-1}{s} ; b_{1}, \ldots, b_{n} ;\left(\frac{s \lambda}{\mu}\right)^{s}\right] . \tag{79}
\end{align*}
$$

Using the properties of the hypergeometric function, we successively obtain that the integral $I$ is

$$
\begin{align*}
I & =\frac{\Gamma(k+1)}{\left(\frac{\bar{n}+1}{\bar{n}}\right)^{k+1}}{ }_{2} F_{1}\left(1, k+1 ; k+1 ; \frac{\bar{n}}{\bar{n}+1}\right)=\frac{\Gamma(k+1)}{\left(\frac{\bar{n}+1}{\bar{n}}\right)^{k+1}}{ }_{1} F_{0}\left(1 ; \frac{\bar{n}}{\bar{n}+1}\right) \\
& =\frac{\Gamma(k+1)}{\left(\frac{\bar{n}+1}{\bar{n}}\right)^{k+1}} \sum_{v=0}^{\infty}\left(\frac{\bar{n}}{\bar{n}+1}\right)^{v}=\frac{\Gamma(k+1)}{\left(\frac{\bar{n}+1}{\bar{n}}\right)^{k+1}}(\bar{n}+1) \tag{80}
\end{align*}
$$

and so, the normalization condition (69) of the $P$-distribution function is accomplished.
Finally, the diagonal representation of the normalized density operator of the PHO in the GK-CSs representation is

$$
\begin{align*}
\rho_{k} & =\frac{1}{\bar{n}}\left(\frac{\bar{n}+1}{\bar{n}}\right)^{k} \int \mathrm{~d} \mu(J, \gamma) \mathrm{e}^{-\frac{1}{n} \frac{J}{2}}|J, \gamma\rangle\langle J, \gamma| \\
& =\left(\mathrm{e}^{2 \beta \hbar \omega_{0}}-1\right)\left(\mathrm{e}^{2 \beta \hbar \omega_{0}}\right)^{k} \int \mathrm{~d} \mu(J, \gamma) \mathrm{e}^{-\left(\mathrm{e}^{2 \beta h \omega_{0}}-1\right) \frac{J}{2}}|J, \gamma\rangle\langle J, \gamma| . \tag{81}
\end{align*}
$$

The diagonal representation of the density operator (or the $P$-representation) is useful for evaluating expectations of the different operators which characterize the PHO quantum system. Then, the thermal expectation value (the thermal average) of an observable $A$ concerning the PHO is given by

$$
\begin{equation*}
\langle A\rangle_{k}=\operatorname{Tr}\left(\rho_{k} A\right)=\int \mathrm{d} \mu(J, \gamma) P_{k}(J)\langle J, \gamma| A|J, \gamma\rangle . \tag{82}
\end{equation*}
$$

If the operator $A$ is diagonal in the $|J, \gamma\rangle$-basis, e.g. if it is an integer power $s$ of the number operator $N$, then, using Eq. (43) and (81), we obtain successively

$$
\begin{align*}
& \left\langle N^{s}\right\rangle_{k}=\frac{1}{\bar{n}}\left(\frac{\bar{n}+1}{\bar{n}}\right)^{k} \int \mathrm{~d} \mu(J, \gamma) \mathrm{e}^{-\frac{1}{\bar{n}} \frac{J}{2}}\langle J, \gamma| N^{s}|J, \gamma\rangle \\
& =\frac{1}{\bar{n}+1}\left(\frac{\mathrm{~d}}{\mathrm{~d} x}\right)^{s}\left(\sum_{v=0}^{\infty} x^{v}\right)=\frac{1}{\bar{n}+1}\left(\frac{\mathrm{~d}}{\mathrm{~d} x}\right)^{s}\left(\frac{1}{1-x}\right) \equiv\left\langle N^{s}\right\rangle . \tag{83}
\end{align*}
$$

For the writing simplification purposes, here we have used the notation $x=\frac{\bar{n}}{\bar{n}+1}$. These expectation values are independent of the Bargmann index $k$.

The thermal expectations of the first and second power of the number operator are

$$
\begin{align*}
& \langle N\rangle=\bar{n}=\frac{1}{\mathrm{e}^{2 \beta \hbar \omega_{0}}-1}  \tag{84}\\
& \left\langle N^{2}\right\rangle=\bar{n}(1+2 \bar{n})=\frac{1}{\mathrm{e}^{2 \beta \hbar \omega_{0}}-1}+2 \frac{1}{\left(\mathrm{e}^{2 \beta \hbar \omega_{0}}-1\right)^{2}} \tag{85}
\end{align*}
$$

Naturally, these expectations are identical with those for the Barut-Girardello coherent states representations [5].

The thermal expectation of the number operator $N$ is the same as the expression of the Bose-Einstein thermal distribution (thermal mean occupancy $\bar{n}$ ). This coincidence suggests that the PHO is suitable for associating it with a boson (e.g. a photon or a phonon).

All in all, we can now define and calculate the thermal second-order correlation function $\left(g^{2}\right)_{k}$ (i.e. the thermal analogue of the second-order correlation function $g_{J}^{2}$ for the state $|J, \gamma\rangle$ )

$$
\begin{equation*}
\left\langle g^{2}\right\rangle_{k} \equiv \frac{\left\langle N^{2}\right\rangle-\langle N\rangle}{\langle N\rangle^{2}}=\left\langle g^{2}\right\rangle=2 \tag{86}
\end{equation*}
$$

and the thermal analogue of the Mandel parameter also $Q_{J}$ [19]:

$$
\begin{equation*}
Q_{k} \equiv\langle N\rangle\left[\left\langle g^{2}\right\rangle-1\right]=\langle N\rangle=\bar{n} . \tag{87}
\end{equation*}
$$

It is not difficult to plot the temperature dependence of these functions.
The normalized density operator characterizes the quantum gas of pseudoharmonic oscillators, regarded as the whole quantum system, is [5]

$$
\begin{equation*}
\rho=\frac{1}{Z} \sum_{j}(2 j+1) Z_{j} \rho_{j}, \tag{88}
\end{equation*}
$$

where $\rho_{j} \equiv \rho_{k}$ is the diagonal representation of the density operator for the rotational state $j$ (see Eq. (81)) and $Z_{j} \equiv Z_{k}$ is the corresponding partition function (see Eq. (65)).

Consequently, the total thermal expectation value of an observable $A$ is

$$
\begin{equation*}
\langle A\rangle=\operatorname{Tr} A \rho=\frac{1}{Z} \sum_{j}(2 j+1) Z_{j} \operatorname{Tr} A \rho_{j} \tag{89}
\end{equation*}
$$

where $\operatorname{Tr} A \rho_{j}=\langle A\rangle_{j}=\langle A\rangle_{k}$ is the expectation value for the rotational state $j$ (see Eq. (82)).
Similarly, the total partition function is

$$
\begin{equation*}
Z=\sum_{j}(2 j+1) \sum_{v} \mathrm{e}^{-\beta E_{v j}}=\sum_{j}(2 j+1) Z_{j}, \tag{90}
\end{equation*}
$$

where $E_{v j} \equiv E_{v k}$ is the energy eigenvalue of the PHO Hamiltonian (see Eq. (10)). Using the total partition function $Z$, we have deduced the expressions of some physical observables which characterize the PHOs quantum canonical gas (e.g. the internal energy, the entropy, the molar heat capacity at the constant volume) [5]. The results we have reached in the quoted paper will not be again repeated in the present one, because they are identical as for the BG-CSs.

## 5. Concluding remarks

Beginning from the observation that the pseudoharmonic oscillator ( PHO ) is an interesting intermediate oscillator model between the three-dimensional harmonic oscillator (HO-3D) and other anharmonic oscillators, in two previous papers [5,15], we have constructed and investigated some properties of the Barut-Girardello coherent states (BG-CSs), respectively of the Klauder-Perelomov coherent states (KP-CSs) associated with the PHO.

In the present paper we have continued our analysis for another kind of coherent states, namely the Gazeau-Klauder coherent states (GK-CSs) of the PHO. Following the standard technique [16,17] we have constructed these states, investigated their properties and calculated some expectation values in the GK-CSs representation. We have comparatively examined some aspects regarding the statistics of all three kinds of CSs associated with the PHO.

The main part of the paper is concentrated on the examination of the statistical properties of the mixed (thermal) states of PHOs quantum canonical gas. We have written the density operator in the GK-CSs representation and, by normalizing it, we have recovered the expression for the partition function $Z_{k}$. The main result of the paper is the construction of the $P$-distribution function in the diagonal representation of the density operator, which allows us to calculate the thermal averages for different physical observables concerning the PHOs quantum canonical gas.

The correct result for the partition function, which we have obtained using the GK-CSs diagonal representation, i.e. the same as in the case of the BG-CSs for the PHO (which was an expected result), allows us to state that all
other expressions for thermal expectation values and physical characteristics (e.g. the internal energy, the entropy and the molar heat capacity) of the PHOs gas are identical with those obtained previously by using the BG-CSs of the PHO [5].

By comparing these coherent states formalisms (especially for the BG-CSs and GK-CSs) of the PHO, we can observe a good parallelism which, naturally, leads to the identification in the case of the thermal values. Consequently, the mathematical facilities brought by each concrete formalism, determine us to use one of them in order to simplify the calculation.

We will observe here that even if the HO-3D can be considered as a limit oscillator of the PHO, it is possible to find a harmonic limit which leads the obtained formulae for the PHO in the GK-CS formalism to the corresponding well-known formulae for the usually Glauber coherent state of the HO-1D. Generally, the GK-CS formalism leads to the usually Glauber coherent state CS formalism of the HO-1D, if we put $z=\sqrt{J} \exp (-\mathrm{i} \gamma)$ (see, the beginning of Sec. 2) [16,17]. In the case of the PHO, due to the double angular frequency confronted by the HO-1D, $\omega=2 \omega_{0}$ (see, Eq. (1)), and the corresponding structure of the quantities $\rho(v)$ (13), we must do the following variable change $z=\sqrt{\frac{J}{2}} \exp (-\mathrm{i} 2 \gamma)$. Consequently, the time-dependent variable for the Eq. (32) is $z(t)=\sqrt{\frac{J}{2}} \exp [-\mathrm{i} 2(\gamma+\omega t)]=$ $\sqrt{\frac{J}{2}} \exp [-\mathrm{i} 2 \gamma(t)]$.

So, we can formally define the following limit (calling it the HO-lD limit):

We will exemplify the calculations of this limit only for three main previously obtained expressions.

1. The definition of the GK-CS of the PHO (12), using Eqs. (14), (10) and (91) leads to the corresponding definition for the Glauber CS of the HO-1D:

$$
\begin{align*}
\lim _{\text {HO-1D }}|J, \gamma\rangle & =\lim _{\text {HO-1D }}\left[\frac{1}{\sqrt{\sum_{v=0}^{\infty} \frac{(1)_{v}}{(k+1)_{v}} \frac{\left(\frac{J}{2}\right)^{v}}{v!}}} \sum_{v=0}^{\infty} \frac{\left(\frac{J}{2}\right)^{v}}{\sqrt{\frac{\Gamma(v+k+1)}{\Gamma(k+1)}}}\left(\mathrm{e}^{-\mathrm{i} 2 \gamma}\right)^{v+k}|v, k\rangle\right] \\
& =\mathrm{e}^{-\frac{|k|^{2}}{2}} \sum_{v=0}^{\infty} \frac{z}{\sqrt{v!}}|v\rangle \equiv|z\rangle . \tag{92}
\end{align*}
$$

2. The Q-function (63) leads to the corresponding Q-function of the HO-1D, i.e.

$$
\begin{align*}
\lim _{\text {HO-1D }}\langle J, \gamma| \rho_{k}|J, \gamma\rangle & =\lim _{\text {HO- } 1 \mathrm{D}}\left[\left(1-\mathrm{e}^{-2 \beta \hbar \omega_{0}}\right) \frac{S_{0}\left(J \frac{\bar{n}}{\bar{n}+1}\right)}{S_{0}(J)}\right] \\
& =\left(1-\mathrm{e}^{-\beta \hbar \omega}\right) \mathrm{e}^{-\left(1-\mathrm{e}^{-\beta \hbar \omega}\right)|z|^{2}} \equiv\langle z| \rho|z\rangle . \tag{93}
\end{align*}
$$

3. The diagonal $P$-function of the density operator (77) has the following limit which is the corresponding $P$-function of the HO-1D:

$$
\begin{align*}
\lim _{\mathrm{HO}-1 \mathrm{D}} P^{(G K)}(J) & =\lim _{\mathrm{HO}-1 \mathrm{D}}\left[\left(\mathrm{e}^{2 \beta \hbar \omega_{0}}-1\right)\left(\mathrm{e}^{2 \beta \hbar \omega_{0}}\right)^{k} \mathrm{e}^{-\left(\mathrm{e}^{2 \beta h \omega_{0}}-1\right) \frac{J}{2}}\right] \\
& =\left(\mathrm{e}^{\beta \hbar \omega}-1\right) \mathrm{e}^{-\left(\mathrm{e}^{\beta \hbar \omega}-1\right)|z|^{2}} \equiv P(|z|) . \tag{94}
\end{align*}
$$

As these three quantities $|J, \gamma\rangle,\langle J, \gamma| \rho_{k}|J, \gamma\rangle$ and $P^{(G K)}(J)$ are fundamental in the whole formalism of the GKCSs, it follows that the HO-1D limit for all results and equations obtained in the present paper for the GK-CSs of the PHO lead to the corresponding results and equations of the usually Glauber CS of the HO-1D. This may be considered as a good suggestion of the correctness of our obtained results.

To sum up, the entire construction and investigation of the Gazeau-Klauder coherent states of the pseudoharmonic oscillator seem to be a completely new approach because, from our knowledge, these results have not yet appeared in the literature. The results demonstrate that apart from their theoretical merit (by contributing to a better understanding of the behavior and properties of the PHO), the formalism of the GK-CSs of the PHO may have also a practical importance (by using it in the quantum information theory and practice).

## Acknowledgement

The present work is performed in part by the financial support through the Grant CNCSIS A647 of the Romanian National Council of Scientific Research. We kindly acknowledge this support.

## References

[1] I.I. Gol'dman, V.D. Krivchenkov, V.I. Kogan, V.M. Galitskii, Problems in Quantum Mechanics, Infosearch, London, 1960.
[2] M. Sage, Chem. Phys. 87 (1984) 431.
[3] M. Sage, M.J. Goodisman, Amer. J. Phys. 53 (1985) 350.
[4] M. Molski, Acta Phys. Polonica A 83 (1993) 417.
[5] D. Popov, J. Phys. A 26 (2001) 1601.
[6] S.H. Dong, Z.Q. Ma, Interat. J. Modern Phys. E 11 (2002) 155.
[7] L.Y. Wang, X.Y. Gu, Z.Q. Ma, S.H. Dong, Found. Phys. Lett. 15 (2002) 569.
[8] A.M. Perelomov, Generalized Coherent states and Their Applications, Springer-Verlag, Berlin, 1986.
[9] C.C. Gerry, J. Math. Phys. 23 (1982) 1995.
[10] J.-P. Antoine, J.-P. Gazeau, P. Monceau, J.R. Klauder, K.A. Penson, J. Math. Phys. 42 (2001) 2349.
[11] A.H.E. Kinani, M. Daoud, Phys. Lett. A 283 (2001) 291.
[12] B. Roy, P. Roy, Phys. Lett. A 296 (2002) 187.
[13] H. Fakhri, A. Chenaghlou, Phys. Lett. A 310 (2003) 1.
[14] D. Popov, Phys. Lett. A 316 (2003) 369.
[15] D. Popov, D.M. Davidovic, D. Arsenovic, V. Sajfert, Acta Phys. Slovaca 56 (2006) 445.
[16] J.-P. Gazeau, J.R. Klauder, J. Phys. A 32 (1999) 123.
[17] J.R. Klauder, K.A. Penson, J.-M. Sixdeniers, Phys. Rev. A 64 (2001) 01381701.
[18] D.F. Walls, G.J. Milburn, Quantum Optics, Springer, Berlin, 1994.
[19] L. Mandel, E. Wolf, Optical Coherence and Quantum Optics, Cambridge University Press, Cambridge, 1995.
[20] K. Fujii, Coherent States and Some Topics in Quantum Information Theory: Review, arXiv:quant-ph/0207178.
[21] X. Wang, B.C. Sanders, S.-H. Pan, J. Phys. A 33 (2000) 7451.
[22] A.P. Proudnikov, Yu.A. Brychkov, O.I. Marichev, Integrals and Series, Elementary Functions, Nauka, Moscow, 1981 (in Russian).
[23] K.A. Penson, A.I. Solomon, J. Math. Phys. 40 (1999) 2354.
[24] J.-M. Sixdeniers, K.A. Penson, A.I. Solomon, J. Phys. A Math. Gen 32 (1999) 7543.
[25] J.-M. Sixdeniers, K.A. Penson, J. Phys. A 33 (2000) 2907.
[26] I.S. Gradshteyn, I.M. Ryzshik, Table of Integrals, Series and Products, 4th ed., Fizmatgiz, Moscow, 1962 (in Russian).
[27] C. Brif, A. Vourdas, A. Mann, J. Phys. A 29 (1996) 5873.
[28] A.M. Mathai, R.K. Saxena, Generalized Hypergeometric Functions with Applications in Statistics and Physical Sciences, in: Lecture Notes in Mathematics, vol. 348, Springer, Berlin, Heidelberg, New York, 1973.
[29] N.I. Akhiezer, The Classical Moment Problem and Some Related Questions in Analysis, Oliver and Boyd, London, 1965.
[30] J. Janszky, A. Gábris, M. Koniorczyk, A. Vukics, P. Adam, Fortschr. Phys. 49 (2001) 993.
[31] S.L. Braunstein, P. van Loock, Rev. Modern Phys. 77 (2005) 513.
[32] S.H. Dong, Factorization Method in Quantum Mechanics, Springer, Dordrecht, 2007.
[33] A.I. Solomon, Phys. Lett. A 196 (1994) 29.
[34] P. Blasiak, K. Penson, A.I. Solomon, Lett. Math. Phys. 67 (2004) 13.
[35] A. Nikiforov, V. Ouvarov, Éléments de la théorie des functions spéciales, Ed. Mir, Moscou, 1976.
[36] E.W. Weisstein, Gamma Distribution, From MathWorld - A Wolfram Web Resource, http://mathworld.wolfram.com/GammaDistribution.html.
[37] D. Popov, Acta Phys. Slovaca $45(1995) 557$.


[^0]:    * Corresponding author.

    E-mail addresses: dusan_popov@yahoo.co.uk (D. Popov), sajfertv@ptt.yu (V. Sajfert), ioan.zaharie@fiz.upt.ro (I. Zaharie).

